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Standard monomials for wonderful group compactifications

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Abstract

Let X be the wonderful compactification of the semisimple adjoint algebraic group G. We show that the basis of $H^0(X, \mathcal{L})$ constructed by Chirivì and Maffei is compatible with all $B \times B$-orbit closures in X by defining subsets using only combinatorics of the underlying paths. Furthermore, we construct standard monomials on X that have properties similar to classical standard monomials.

Keywords: Standard monomials; Group compactification; Wonderful

1. Introduction

Let G be a semisimple algebraic group over an algebraically closed field. One aim of standard monomial theory is to define bases of the space of global sections $H^0(G/B, \mathcal{L}_\lambda)$. Here B is a Borel subgroup of G, G/B is the flag variety, and λ is a dominant weight of G with associated line bundle $\mathcal{L}_\lambda$. The elements of this basis should be weight vectors of G and behave nicely under restriction to the Schubert varieties in G/B. As $H^0(G/B, \mathcal{L}_\lambda) \cong V(\lambda)^*$ by the Borel–Weil theorem, this also gives bases of the highest weight modules of G that have nice geometric properties.

One solution for this is the path model which defines paths in the weight lattice and an associated path vector $p_\pi \in H^0(G/B, \mathcal{L}_\lambda)$ for every LS-path of the form λ. These path vectors form a basis of the G-module $H^0(G/B, \mathcal{L}_\lambda)$. Given a Schubert variety $S(w)$ in G/B, it is possible

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to define a subset of the set of all LS-paths of the form λ such that the associated path vectors restrict to a basis of $H^0(S(w), \mathcal{L}_\lambda)$. This definition depends purely on combinatorial properties of the paths. The path vectors associated to LS-paths are called standard monomials an G/B.

As a generalization, it would be nice to have an analogue for compactifications of symmetric spaces G/H instead of the flag variety G/B. This paper deals with the special case of the wonderful compactification of an adjoint group G, that is considered as a symmetric space of the group $G \times G$. The wonderful compactification X of the group G consists of several $G \times G$-orbits of which exactly one is closed, and this unique closed orbit Y is isomorphic to $G/B \times G/B$. Now the existing standard monomials on $G/B \times G/B$ can be extended to X to get standard monomials on X.

In this paper we show, that arbitrary extensions are compatible not only with the closures of $G \times G$-orbits in X, but also with the closures of all $B \times B$-orbits. Furthermore, we construct extensions that possess attributes similar to those of classical standard monomials.

The paper is organized as follows. In the second section, we shortly recall the construction of the wonderful compactification X of a group G. We also include the description of the $B \times B$-orbits in X and their closure relations obtained by Springer. Furthermore, a short description of the line bundles on X is given.

In the third section, we very briefly describe standard monomials on the flag variety G/B. We do not give an adequate presentation of the path model and standard monomials, but state only those properties of LS-paths and path vectors we will need later.

The fourth section contains a resume of the results of the paper [5], in which Chirivì and Maffei construct standard monomials for wonderful compactifications of symmetric spaces. We give the definition and the properties in the special case of the wonderful compactification of a group. In particular, the set $\mathcal{M}^{(\lambda)}$ is defined which is a basis for $H^0(X, \mathcal{L}_\lambda)$ compatible with restrictions to $G \times G$-orbit closures.

In the fifth section, we define for every $B \times B$-orbit closure Z a subset $\mathcal{M}_Z^{(\lambda)}$ of $\mathcal{M}^{(\lambda)}$ which restricts to a linearly independent subset of $H^0(Z, \mathcal{L}_\lambda)$. Using consequences of the existence of a compatible Frobenius splitting and properties of the associated graded module for a suitable filtration, we prove that $\mathcal{M}_Z^{(\lambda)}$ is also a basis of the global sections on Z. Thus, we show that in the case of a group compactification the basis $\mathcal{M}^{(\lambda)}$ is compatible with restrictions to $B \times B$-orbit closures as well.

In the last section we take advantage of the fact that so far, none of the constructions depends on the choice of the continuation of the standard monomials on Y to X. This section is dedicated to the construction of standard monomials for X which have properties similar to the classical standard monomials.

2. Wonderful group compactifications

In [6], De Concini and Procesi construct the wonderful compactification of an adjoint group G in characteristic zero as a special case of symmetric spaces. The definition was extended to positive characteristic by Strickland in [13]. In this section, we shortly recall the construction and properties of wonderful group compactifications.

Let G be an adjoint semisimple algebraic group over an algebraically closed field k of arbitrary characteristic. Choose a Borel subgroup B and a torus T such that $T \subset B \subset G$. The corresponding weight lattice is Λ, with dominant weights Λ^+. The set of simple roots is $\Delta = \{\alpha_1, \dots, \alpha_l\}$, the positive roots are denoted by Φ^+ and the negative roots by $\Phi^- = -\Phi^+$.

Let $\tilde{G}$ be the simply connected covering of G with morphism $\pi_{\mathrm{ad}}: \tilde{G} \to G$. Denote the pre-images of the subgroups of G in $\tilde{G}$ by $\tilde{T} = \pi_{\mathrm{ad}}^{-1}(T)$, $\tilde{B} = \pi_{\mathrm{ad}}^{-1}(B)$.

The group G can be considered as the symmetric space $(G \times G)/\mathrm{diag}_G$, where diag_G denotes the diagonal in $G \times G$. For a suitable G-module M, this space $G \cong (G \times G)/\mathrm{diag}_G$ is isomorphic to the orbit $(G \times G) \cdot h$ of an element $h \in \mathbb{P}(\mathrm{End}(M))$, where $G \times G$ acts on $\mathrm{End}(M) \cong M^* \otimes M$ via $(g_1, g_2) \cdot (m_1 \otimes m_2) = g_1 m_1 \otimes g_2 m_2$. Thus, G is embedded in $\mathbb{P}(\mathrm{End}(M))$. Define the wonderful compactification X of G as the closure of $(G \times G) \cdot h$ in $\mathbb{P}(\mathrm{End}(M))$. In characteristic zero, M can be taken as the highest weight module $V(\lambda)$ for any regular weight λ. The wonderful compactification will be independent of the chosen weight λ. In positive characteristic, the Steinberg module is such a suitable module.

Proposition 1. (Theorem 3.1 in [6]) *Let X be the wonderful compactification of an adjoint group G of rank l.*

(1) *X is smooth.*
(2) *$X \setminus (G \times G) \cdot h$ is the union of l smooth divisors $S_1, \ldots, S_l$, which intersect transversally.*
(3) *There is a bijection between the subsets of $D := \{1, 2, \ldots, l\}$ and the set of $G \times G$-orbits in X given by $I \mapsto X_I^\circ$ where the closure of X_I° is $X_I := \bigcap_{i \notin I} S_i$.*
(4) *The unique closed $G \times G$-orbit $X_\emptyset = \bigcap_{i=1}^{l} S_l$ is isomorphic to $G/B \times G/B$ and also denoted by Y.*

So for two subsets $I, J \subset D$ one has $I \subset J \Leftrightarrow X_I \subset X_J$. Every $G \times G$-orbits contains a *base point* h_I, which has the following properties:

(1) $(B \times B^-) \cdot h_I$ is dense in X_I, and
(2) there is a cocharacter γ of T such that $h_I = \lim_{t \to 0} \gamma(t)$.

Springer also describes the $B \times B$-orbits and their closure relations explicitly.

Proposition 2. (Lemma 1.3 in [12])

(1) *The $B \times B$-orbits in X are*

$$[I, x, w] := (B \times B) \cdot (x, w) \cdot h_I,$$

where $I \subseteq D$, $x \in W^I$, and $w \in W$. Here W is the Weyl group of G, W_I the parabolic subgroup of W generated by the simple reflections corresponding to the roots α_i such that $i \in I$, and W^I is the set of minimal representatives in W/W_I.
(2) *The relation between the set $\{(I, x, w) \mid I \subseteq D,\ x \in W^I,\ w \in W\}$ and the set of $B \times B$-orbits in X is a bijection.*
(3) *The dimension of a $B \times B$-orbit is given by*

$$\dim[I, x, w] = l(w_0) - l(x) + l(w) + |I|.$$

Here w_0 is the longest element in W and $l(w)$ denotes the length of the element w of the Weyl group.

There is a partial order on the set of $B \times B$-orbits defined by

$$[I, x', w'] \leqslant [J, x, w] \quad :\Leftrightarrow \quad \overline{[I, x', w']} \subseteq \overline{[J, x, w]}.$$

The closures of the $B \times B$-orbits $[\emptyset, x, w]$, where $x, w \in W$, are the well-known Schubert varieties in $Y \cong G/B \times G/B$. Denoting the Schubert variety in G/B corresponding to the Weyl group element w by $S(w)$, one has

$$\overline{[\emptyset, x, w]} \cong S(xw_0) \times S(w).$$

The closures of the $B \times B$-orbits $[D, \mathrm{id}, w]$ where $w \in W$ are called *large Schubert varieties* and denoted by $X(w)$.

Proposition 3. (Proposition 2.4 in [12]) *Let $[I, x', w']$, $[J, x, w]$ be two $B \times B$-orbits in X where $I, J \subset D$, $x' \in W^I$, $x \in W^J$, and $w', w \in W$. The relation $[I, x', w'] \leqslant [J, x, w]$ holds if and only if*

(1) *$I \subseteq J$, and*
(2) *there exist $u \in W_I$ and $v \in W_J \cap W^I$ such that*
 (a) *$l(wv) = l(w) + l(v)$,*
 (b) *$x' \geqslant xvu^{-1}$, and*
 (c) *$w'u \leqslant wv$.*

Here $\leqslant$ denotes the Bruhat order on the Weyl group.

Proposition 4. (Lemma 2.4 in [13]) *The restriction map* $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *is injective. Its image is the sub-lattice* $\{(-w_0\lambda, \lambda) \mid \lambda \in \Lambda\} \cong \Lambda$.

So we get $\Lambda \cong \mathrm{Pic}(X)$ via $\lambda \mapsto \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda$ is the line bundle on X which restricts to $\mathcal{L}^{(G/B)}_{-w_0\lambda} \boxtimes \mathcal{L}^{(G/B)}_\lambda$ on $Y \cong G/B \times G/B$. Here $\mathcal{L}^{(G/B)}_\lambda$ denotes the line bundle $G \times_B k_{-\lambda}$ on G/B associated to the weight λ.

Proposition 5. (Corollary 8.2 in [6] and Section 1 in [4]) *For every $i = 1, \dots, l$ there is a $G \times G$-invariant section $\sigma_i \in H^0(X, \mathcal{L}_{\alpha_i})$ with divisor S_i, that is unique up to multiplication by a scalar.*

Let $\lambda, \mu \in \Lambda$ be two weights. Write $\mu \leqslant \lambda$ if there are non-negative numbers $n_1, \dots, n_l \in \mathbb{N}_0$ such that $\lambda - \mu = \sum_{i=1}^l n_i\alpha_i$. Denote such a collection of numbers by

$$\vec{n} = (n_1, \dots, n_l) \in \mathbb{N}_0^l.$$

The norm of the vector $\vec{n}$ is $|\vec{n}| = \sum_{i=1}^l n_i$. Denoting $\vec{\alpha} = (\alpha_1, \dots, \alpha_l)$, the scalar product is $\vec{n}\vec{\alpha} = \sum_{i=1}^l n_i\alpha_i$. There are two convenient ways to refer to products of σ_i. Write

$$\sigma^{\vec{n}} = \sigma_1^{n_1} \cdots \sigma_l^{n_l} = \sigma^{(\lambda-\mu)}$$

in case $\mu \leqslant \lambda$ and $\lambda - \mu = \vec{n}\vec{\alpha}$. Write in this case also $|\lambda - \mu| = |\vec{n}|$.

Proposition 6. (Theorem 8.3 in [6]) *Let* $\lambda \in \Lambda$.

$$H^0(X, \mathcal{L}_\lambda) \neq 0 \quad \Leftrightarrow \quad \exists \mu \in \Lambda^+ \text{ such that } \mu \leqslant \lambda.$$

3. Standard monomials for G/B

In this section, we briefly recall the facts we will need about the path model and standard monomials. We refer to [10] for a more detailed exposition.

One of the basic results of standard monomial theory is a set $\{p_\pi^{(\lambda)} \mid \pi \in B_\lambda\}$ which forms a basis of the G-module $H^0(G/B, \mathcal{L}_\lambda)$ and has some nice properties. Its elements $p_\pi^{(\lambda)}$ are called *path vectors* and are indexed by the set B_λ of LS-paths π of the form $\lambda \in \Lambda^+$. Two of their properties will be of importance later on. First, the end point $\pi(1)$ of a path $\pi \in B_\lambda$ is in Λ, and the corresponding path vector $p_\pi^{(\lambda)}$ is a weight vector of weight $-\pi(1)$. The second property is that there exists a map $i : B_\lambda \to W$ which assigns to an LS-path π an element of the Weyl group $i(\pi)$ called its initial direction. This is used to define for a path the notion of being standard on a Schubert variety in G/B.

Definition 7. Let $Y = \bigcup X(\tau_i)$ be a union of Schubert varieties $X(\tau_i)$ in G/B. The LS-path $\pi \in B_\lambda$ is *standard on* Y if and only if $i(\pi) \leqslant \tau_i$ for at least one τ_i. Here $i(\pi) \in W$ is the initial direction of the path π and $\leqslant$ denotes the Bruhat order on the Weyl group. The associated path vector $p_\pi \in H^0(G/B, \mathcal{L}_\lambda)$ is *standard on* Y if and only if the path π is.

Proposition 8. (Theorem 5.3, Corollary 5.2, Theorem 8.6 in [10])

(1) *The set of path vectors* $\{p_\pi \mid \pi \in B_\lambda\}$ *of the form* λ *is a basis of* $H^0(G/B, \mathcal{L}_\lambda)$.
(2) $\{p_\pi|_Y \mid \pi \in B_\lambda,\ \pi \text{ standard on } Y\}$ *is a basis of* $H^0(Y, \mathcal{L}_\lambda|_Y)$.
(3) $\{p_\pi \mid \pi \in B_\lambda,\ \pi \text{ not standard on } Y\}$ *is a basis of the kernel of the restriction map* $H^0(G/B, \mathcal{L}_\lambda) \to H^0(Y, \mathcal{L}_\lambda|_Y)$.

4. Standard monomials for X

In [5], Chirivì and Maffei construct standard monomials for the wonderful embedding of a symmetric space G/H that extend the classical standard monomials. The construction and properties of the extended monomials in the case $X = \bar{G}$ are given in this section.

Let X be the wonderful compactification of the group G and $\lambda \in \Lambda^+$ a dominant weight. The line bundle $\mathcal{L}_\lambda \in \mathrm{Pic}(X)$ can be $\tilde{G} \times \tilde{G}$-linearized. This gives a linear action of $\tilde{G} \times \tilde{G}$ on $H^0(X, \mathcal{L}_\lambda)$. For any closed $G \times G$-stable subvariety Z of X, we consider $H^0(Z, \mathcal{L}_\lambda)$ as a $\tilde{G} \times \tilde{G}$-module.

The following proposition is proved by De Concini and Procesi in the proof of Theorem 8.3 in [6] in case $k = \mathbb{C}$. The arguments used are generalized to arbitrary characteristic by De Concini and Springer in the last paragraph of [7].

Proposition 9. *For all* $\lambda \in \Lambda^+$ *the restriction map* $H^0(X, \mathcal{L}_\lambda) \to H^0(Y, \mathcal{L}_\lambda)$ *is surjective.*

Let $\lambda \in \Lambda^+$ be a dominant weight. For any dominant $\mu \leqslant \lambda$ and any LS-path $\pi \in B_\mu$ there exists a path vector $p_\pi^{(\mu)} \in H^0(Y, \mathcal{L}_\mu)$. Choose for every $p_\pi^{(\mu)}$ an arbitrary continuation $x_\pi^{(\mu)} \in H^0(X, \mathcal{L}_\mu)$ such that $x_\pi^{(\mu)}|_Y = p_\pi^{(\mu)}$.

Proposition 10. (Theorem 3.3 in [5]) *The set*

$$\mathcal{M}^{(\lambda)} := \left\{\sigma^{(\lambda-\mu)} x_\pi^{(\mu)} \mid \mu \in \Lambda^+,\ \mu \leqslant \lambda,\ \pi \in B_\mu\right\}$$

is a basis of $H^0(X, \mathcal{L}_\lambda)$.

The next proposition shows that this basis is compatible with the $G \times G$-orbits in X.

The closure X_I of the $G \times G$-orbit corresponding to $I \subseteq D$ satisfies $X_I = \bigcap_{i \notin I} S_i$. Hence, the restriction of the $G \times G$-invariant section σ_i to X_I is non-zero if and only if $i \in I$. In particular, on the unique closed orbit $X_\emptyset = Y$ we have $\sigma_1|_Y = \cdots = \sigma_l|_Y = 0$.

Definition 11. Let X_I be the closure of a $G \times G$-orbit in X and $\lambda, \mu \in \Lambda^+$ two dominant weights such that $\mu \leqslant \lambda$. Then $\lambda - \mu = \sum_{i=1}^{l} n_i \alpha_i$ where $n_i \in \mathbb{N}_0$ for all $i \in D$. The monomial $\sigma^{(\lambda-\mu)} x_\pi^{(\mu)} \in \mathcal{M}^{(\lambda)}$ is *standard on* X_I if and only if $n_i = 0$ for all $i \notin I$.

Proposition 12. (Corollary 3.4 in [5]) *Let* X_I *be the closure of a* $G \times G$*-orbit in* X*. The set*

$$\mathcal{M}_{X_I}^{(\lambda)} := \left\{\sigma^{(\lambda-\mu)} x_\pi^{(\mu)}\big|_{X_I} \mid \sigma^{(\lambda-\mu)} x_\pi^{(\mu)} \textit{ standard on } X_I\right\}$$

is a basis of $H^0(X_I, \mathcal{L}_\lambda)$.

5. A basis for $H^0(Z, \mathcal{L}_\lambda)$

Let X be the wonderful compactification of the group G and Z the closure of a $B \times B$-orbit in X. In this part, for any $\lambda \in \Lambda^+$ we define a subset $\mathcal{M}_Z^{(\lambda)}$ of $\mathcal{M}^{(\lambda)}$ which is a basis of the $\tilde{B} \times \tilde{B}$-module $H^0(Z, \mathcal{L}_\lambda)$.

Consider a $B \times B$-orbit $[I, x, w]$, $I \subseteq D$, $x \in W^I$, $w \in W$, and its closure Z. As the intersection $Z \cap Y$ with the unique closed $G \times G$-orbit $Y = X_\emptyset$ is closed and $B \times B$-stable, it is a union of Schubert varieties in $Y \cong G/B \times G/B$. A basis of $H^0(Z \cap Y, \mathcal{L}_\mu|_{Z\cap Y})$ for $\mu \in \Lambda^+$ is given by the set of restrictions of the standard monomials $p_\pi^{(\mu)}|_{Z\cap Y}$, that are standard on $Z \cap Y$.

For any dominant weight $\mu \in \Lambda^+$ and any LS-path $\pi \in B_\mu$ let $x_\pi^{(\mu)} \in H^0(X, \mathcal{L}_\mu)$ be an arbitrary continuation of the standard monomial $p_\pi^{(\mu)} \in H^0(Y, \mathcal{L}_\mu)$ to X.

Definition 13. The path $\pi \in B_\mu$ and $x_\pi^{(\mu)}$ are *standard on* Z if and only if $p_\pi^{(\mu)} = x_\pi^{(\mu)}|_Y$ is standard on the union of Schubert varieties $Z \cap Y$.

Proposition 14. *Let* Z *be the closure of the* $B \times B$*-orbit* $[I, x, w]$ *in* X*, where* $I = \{i_1, \ldots, i_r\} \subseteq D$*,* $x \in W^I$*, and* $w \in W$*.*

$$\mathcal{M}_Z^{(\lambda)} := \left\{\sigma_{i_1}^{n_1} \cdots \sigma_{i_r}^{n_r} x_\pi^{(\mu)}\big|_Z \;\middle|\; \mu = \lambda - \sum_{k=1}^{r} n_k \alpha_{i_k} \in \Lambda^+,\ n_1, \ldots, n_r \in \mathbb{N}_0,\ \pi \in B_\mu \textit{ standard on } Z\right\}$$

is a linearly independent subset of $H^0(Z, \mathcal{L}_\lambda|_Z)$.

Proof. Consider the equation

$$\sum_{\mu \leqslant \lambda} \sum_{\pi \in B_\mu} \beta_\pi^{(\mu)} \sigma_{i_1}^{n_1} \cdots \sigma_{i_r}^{n_r} x_\pi^{(\mu)}\big|_Z = 0, \tag{1}$$

where $\beta_\pi^{(\mu)} \in k$ and $x_\pi^{(\mu)}$ is standard on Z. Here the sum is over all $\mu = \lambda - \sum_{k=1}^r n_k \alpha_{i_k} \in \Lambda^+$ such that $n_1, \ldots, n_r \in \mathbb{N}_0$.

All σ_i are zero on the closed $G \times G$-orbit Y, so the restriction of (1) to $Z \cap Y$ yields

$$\sum_{\pi \in B_\lambda} \beta_\pi^{(\lambda)} x_\pi^{(\lambda)}\big|_{Z \cap Y} = 0.$$

But $p_\pi^{(\lambda)} = x_\pi^{(\lambda)}|_Y$ are standard and in particular linearly independent on $Z \cap Y$, so $\beta_\pi^{(\lambda)} = 0$ for all $\pi \in B_\lambda$.

The next aim is to show that Eq. (1) also implies $\beta_\pi^{(\mu)} = 0$ for all $\pi \in B_\mu$ and $\mu \in M := \{\mu \leqslant \lambda \mid \mu = \lambda - \sum_{k=1}^r n_k \alpha_{i_k} \in \Lambda^+,\ n_k \in \mathbb{N}_0\}$, $\mu \neq \lambda$. Therefore, a lexicographic order on the set $\{\mu \in \Lambda^+ \mid \mu \leqslant \lambda\}$ is defined as follows:

Let $\mu, \mu' \leqslant \lambda$ where $\mu = \lambda - \sum_{i=1}^l n_i \alpha_i \in \Lambda^+$, $\mu' = \lambda - \sum_{i=1}^l n_i' \alpha_i \in \Lambda^+$. We have $\mu >_{\text{lex}} \mu'$ if and only if there exists $j \in \{1, \ldots, l\}$ such that $n_i = n_i'$ for all $i < j$ and $n_j < n_j'$.

This definition yields a total ordering on the set $\{\mu \in \Lambda^+ \mid \mu \leqslant \lambda\}$ which can be restricted to the subset M that contains those $\mu = \lambda - \sum_{j=1}^l n_j \alpha_j \in \Lambda^+$ where $n_j = 0$ for $j \notin I = \{i_1, \ldots, i_r\}$.

Now consider a weight $\nu \in M$, $\nu < \lambda$ and assume $\beta_\pi^{(\mu)} = 0$ for all $\pi \in B_\mu$ and $\mu >_{\text{lex}} \nu$. It remains to show that $\beta_\pi^{(\nu)} = 0$ for all $\pi \in B_\nu$. Let $\lambda - \nu = \sum_{k=1}^r n_k \alpha_{i_k}$ and $j \in \{1, \ldots, r\}$ such that $n_j \neq 0$ and $n_k = 0$ for all $k < j$.

On the closure $X_{\{\alpha_{i_j}, \ldots, \alpha_{i_r}\}}$ of the $G \times G$-orbit $X^\circ_{\{\alpha_{i_j}, \ldots, \alpha_{i_r}\}}$ we have $\sigma_{i_1} = \cdots = \sigma_{i_{j-1}} = 0$. So if we restrict the sections, Eq. (1) becomes

$$\sum_{\substack{(m_j, \ldots, m_r) \in \mathbb{N}^{r-j+1} \\ \mu = \lambda - \sum_{k=j}^r m_k \alpha_{i_k} \in \Lambda^+}} \sum_{\pi \in B_\mu} \beta_\pi^{(\mu)} \sigma_{i_j}^{m_j} \cdots \sigma_{i_r}^{m_r} x_\pi^{(\mu)}\big|_{Z \cap X_{\{\alpha_{i_j}, \ldots, \alpha_{i_r}\}}} = 0$$

where $m_j \geqslant n_j$. All $\sigma_{i_j}^{m_j} \cdots \sigma_{i_r}^{m_r} x_\pi^{(\mu)}|_{Z \cap X_{\{\alpha_{i_j}, \ldots, \alpha_{i_r}\}}}$ lie in the image of

$$H^0(Z \cap X_{\{\alpha_{i_j}, \ldots, \alpha_{i_r}\}}, \mathcal{L}_{\lambda - n_j \alpha_{i_j}}) \xrightarrow{\sigma_{i_j}^{n_j} \cdot} H^0(Z \cap X_{\{\alpha_{i_j}, \ldots, \alpha_{i_r}\}}, \mathcal{L}_\lambda).$$

The wonderful compactification of a group is in particular a complete regular $G \times G$-variety, so Theorem 1.4 in [2] can be applied. Part (ii) implies that the intersection of an irreducible component of $Z \cap X_J$ with the $G \times G$-orbit X_J° for $J \subseteq I$ is non-empty. Since σ_{i_j} is $G \times G$-invariant and does not vanish on the $G \times G$-orbit X_J° for $\alpha_{i_j} \in J$, the multiplication by $\sigma_{i_j}^{n_j}$ is an injective map. Therefore, the set $\{\sigma_{i_j}^{m_j}\cdots\sigma_{i_r}^{m_r}x_\pi^{(\mu)}|_{Z\cap X_{\{\alpha_{i_j},\dots,\alpha_{i_r}\}}}\}$ is linearly independent if and only if the set of pre-images

$$\left\{\sigma_{i_j}^{m_j-n_j}\sigma_{i_{j+1}}^{m_{j+1}}\cdots\sigma_{i_r}^{m_r}x_\pi^{(\mu)}\big|_{Z\cap X_{\{\alpha_{i_j},\dots,\alpha_{i_r}\}}}\right\}\subseteq H^0(Z\cap X_{\{\alpha_{i_j},\dots,\alpha_{i_r}\}},\mathcal{L}_{\lambda-n_j\alpha_{i_j}})$$

is, too. Restricting to the closure of the $G\times G$-orbit $X_{\{\alpha_{i_{j+1}},\dots\alpha_{i_r}\}}$ the equation

$$\sum_{\substack{(m_j,\dots,m_r)\in\mathbb{N}^{r-j+1}\\ \mu=\lambda-\sum_{k=j}^r m_k\alpha_{i_k}\in\Lambda^+}}\sum_{\pi\in B_\mu}\beta_\pi^{(\mu)}\sigma_{i_j}^{m_j-n_j}\sigma_{i_{j+1}}^{m_{j+1}}\cdots\sigma_{i_r}^{m_r}x_\pi^{(\mu)}\big|_{Z\cap X_{\{\alpha_{i_j},\dots,\alpha_{i_r}\}}}=0$$

becomes

$$\sum_{\substack{(m_{j+1},\dots,m_r)\in\mathbb{N}^{r-j}\\ \mu=\lambda-n_j\alpha_{i_j}-\sum_{k=j+1}^r m_k\alpha_{i_k}\in\Lambda^+}}\sum_{\pi\in B_\mu}\beta_\pi^{(\mu)}\sigma_{i_{j+1}}^{m_{j+1}}\cdots\sigma_{i_r}^{m_r}x_\pi^{(\mu)}\big|_{Z\cap X_{\{\alpha_{i_{j+1}},\dots,\alpha_{i_r}\}}}=0,$$

where $m_{j+1}\geqslant n_{j+1}$ by induction hypothesis. Hence, the last steps can be repeated with $j+1$, $j+2,\dots,r$. Finally, Eq. (1) gives

$$\sum_{\pi\in B_\nu}\beta_\pi^{(\nu)}x_\pi^{(\nu)}\big|_{Z\cap Y}=0.$$

As all $x_\pi^{(\nu)}$ are standard on $Z\cap Y$, this implies $\beta_\pi^{(\nu)}=0$ for all $\pi\in B_\nu$. $\square$

Remark 15. The same proof also works for corresponding sets when $X=\overline{G/H}$ is the wonderful compactification of an arbitrary symmetric space (see Proposition 2.1 in [1]). In this case, standard monomials on the closure Z of a B-orbit can be defined in the same way. But while the set $\mathcal{M}_Z^{(\lambda)}$ is still linearly independent, in general it is not a basis of $H^0(Z,\mathcal{L}_\lambda)$.

To show that $\mathcal{M}_Z^{(\lambda)}$ is a basis, the dimension of the $\tilde{B}\times\tilde{B}$-module $H^0(Z,\mathcal{L}_\lambda)$ has to be calculated. For this, we follow the approach of Brion and Polo in [4] and generalize some of their results on large Schubert varieties to arbitrary $B\times B$-orbit closures.

First, we need some facts that are obtained by using the methods of Frobenius splitting. For this, we recall some important definitions.

Definition 16. (Definitions 1.1 and 1.2 in [11]) Let X be a projective variety over an algebraically closed field k of characteristic $p>0$ and Y a closed subvariety with ideal sheaf $\mathcal{I}$. The absolute Frobenius morphism is denoted by $F:X\to X$ as well as the pth power map $F:\mathcal{O}_X\to F_*\mathcal{O}_X$. Let $\mathcal{L}$ be a line bundle on X and $s:\mathcal{O}_X\to\mathcal{L}$ be a non-zero section of $\mathcal{L}$ with zeroes precisely on the divisor D.

(1) X is *Frobenius split* if there is an $\mathcal{O}_X$-module morphism $\varphi:F_*\mathcal{O}_X\to\mathcal{O}_X$ such that $\varphi\circ F=\mathrm{id}_{\mathcal{O}_X}$. The morphism φ is called *Frobenius splitting*.
(2) The Frobenius splitting φ splits X *compatibly with* Y if $\varphi(F_*\mathcal{I})=\mathcal{I}$.

(3) X is *Frobenius D-split* if there is an $\mathcal{O}_X$-linear map $\Psi : F_*\mathcal{L} \to \mathcal{O}_X$ such that $\varphi = \Psi \circ F_*(s)$ is a Frobenius splitting of X.
(4) Y is *compatibly D-split* if $\varphi = \Psi \circ F_*(s)$ splits X compatibly with Y and no irreducible component of Y is contained in $\operatorname{supp}(D)$.

Let the algebraic group G be defined over an algebraically closed field of positive characteristic. In [4], Brion and Polo construct a Frobenius splitting $\sigma \in \operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ that splits X compatibly with the closures of all $G \times G$-orbits X_I and with the large Schubert varieties $X(w)$ (Theorem 2 in [4]). He and Thomsen show in [9] that any splitting, that splits X compatibly with these subvarieties, splits X compatibly also with any $B \times B$-orbit closure Z (see the proof of Proposition 7.1 in [9]), because every such Z is an irreducible component of the intersection of two $B \times B$-orbit closures of higher dimension (Proposition 6.5 in [9]). Combining this, we have

Proposition 17. *The splitting σ constructed by Brion and Polo in Theorem* 2 *in* [4] *splits X compatibly with all closures of $B \times B$-orbits in X.*

Corollary 18. *Let X be the wonderful compactification of the adjoint group G over an algebraically closed field of arbitrary characteristic. Let $\lambda \in \Lambda^+$, X_I be the closure of a $G \times G$-orbit and Z the closure of a $B \times B$-orbit in X.*

(1) *The restriction maps* $\operatorname{res}_Z : H^0(X, \mathcal{L}_\lambda) \to H^0(Z, \mathcal{L}_\lambda)$ *and* $\operatorname{res}_{Z\cap Y} : H^0(Z, \mathcal{L}_\lambda) \to H^0(Z \cap Y, \mathcal{L}_\lambda)$ *are surjective. Here $\mathcal{L}_\lambda$ also denotes the restriction of the line bundle to Z respectively $Z \cap Y$.*
Furthermore, $H^i(Z, \mathcal{L}_\lambda) = 0$ for all $i > 0$.
(2) *The scheme theoretic intersection $Z \cap X_I$ is reduced.*

Proof. First, consider the case $\operatorname{char}(k) = p > 0$. If λ is regular, then the line bundle $\mathcal{L}_\lambda$ is ample by Lemma 1 in [4] and the first assertion is just Proposition 7.2 in [9] for the special case of the wonderful compactification. To prove the assertion in case λ is not regular, we will use Proposition 1.13(ii) from [11]. To apply this proposition, we have to show the following.

(1) There is a divisor D such that X is Frobenius D-split and the corresponding line bundle $\mathcal{L} = \mathcal{O}(D)$ is ample,
(2) any $B \times B$-orbit closure Z is compatibly D-split, and
(3) $\mathcal{L}_\lambda$ is without base points.

In the proof of Theorem 2 in [4] Brion and Polo show, that σ splits X compatibly with the $B \times B$-stable divisor $D^+ := \sum_{i=1}^{l} X(w_0 s_i)$ as well as the $B^- \times B^-$-stable divisor $D^- := (w_0, w_0)D^+ = \sum_{i=1}^{l} X^-(s_i w_0)$. Here B^- denotes the opposite Borel of B and $X^-(w) = \overline{B^- w B^-}$, where $w \in W$, is an opposite large Schubert variety in X. In particular, by Theorem 1.4.10 in [3] σ is a $(p-1)D^-$-splitting.

Furthermore, in the proof of Theorem 2 in [4] it is shown, that the line bundle associated to the divisor $(p-1)D^-$ is $\mathcal{L}_{(p-1)\varrho}$. As ϱ is a regular weight, this line bundle is ample again by Lemma 1 in [4]. So, setting $D := (p-1)D^-$, we get (1).

The support $T := \operatorname{supp}(D) = \operatorname{supp}((p-1)D^-) = \bigcup_{i=1}^{l} X^-(s_i w_0)$ contains no $B \times B$-orbit. Indeed, if $x \in T$ such that $(B \times B)x \subseteq T$, then $(B^- \times B^-)(B \times B)x \subseteq T$, because T is $B^- \times B^-$-stable. As $B^- B$ is dense in G and T is closed, this implies $(G \times G)x \subseteq T$. But this is not possible, because T contains no $G \times G$-orbit. Hence, no irreducible component of any closed union of $B \times B$-orbits is contained in supp(D). Combined with the fact that σ splits X compatibly with all considered subvarieties Z this yields (2).

For dominant λ, the line bundle $\mathcal{L}_\lambda$ is generated by its global sections and therefore without base points by Lemma 1 in [4]. This shows (3).

So Proposition 1.13(ii) in [11] can be applied and yields the first assertion in positive characteristic. Using Proposition 1.6.2 and Corollary 1.6.3 in [3], this implies the same result in characteristic zero.

The second assertion is an easy consequence of the existence of a splitting in positive characteristic and can be found for example in Proposition 1.2.1 in [3]. Corollary 1.6.6 in [3] extends the result to characteristic zero. □

Recall that Z is the closure of the $B \times B$-orbit $[I, x, w]$ where $I = \{i_1, \dots, i_r\} \subseteq D$, $x \in W^I$, and $w \in W$.

Lemma 19. *Let $J \subseteq I$ and X_J be the corresponding $G \times G$-orbit closure. The irreducible components of $Z \cap X_J$ are $\overline{[J, xv, wv]}$ where $v \in W_I \cap W^J$ such that $l(wv) = l(w) + l(v)$.*

Proof. Part (ii) of Theorem 1.4 in [2] implies that every irreducible component meets the $G \times G$-orbit X_J°, so $Z \cap X_J$ is the union of all $\overline{[J, \tilde{x}, \tilde{w}]}$ such that $[J, \tilde{x}, \tilde{w}] \leqslant [I, x, w]$,

$$\begin{aligned}
[J, \tilde{x}, \tilde{w}] \leqslant [I, x, w] \quad &\Leftrightarrow \quad \exists v \in W_I \cap W^J \text{ such that } l(wv) = l(w) + l(v) \text{ and} \\
& \quad \exists u \in W_J \text{ such that } \tilde{x} \geqslant xvu^{-1} \text{ and } \tilde{w}u \leqslant wv \\
&\Leftrightarrow \quad \exists v \in W_I \cap W^J \text{ such that } l(wv) = l(w) + l(v) \text{ and} \\
& \quad [J, \tilde{x}, \tilde{w}] \leqslant [J, xv, wv].
\end{aligned}$$

Hence, those $\overline{[J, xv, wv]}$ are the irreducible components. □

Lemma 20. *Let $\mu \in \Lambda$ be a weight,*

$$\mu \notin \bigcap_{i \in I} \alpha_i^+ \quad \Rightarrow \quad H^0(Z \cap Y, \mathcal{L}_\mu|_{Z \cap Y}) = 0.$$

Here for a simple root α, denote by α^+ the set $\{\lambda \in \Lambda \mid \langle \lambda, \check{\alpha} \rangle \geqslant 0\}$.

Proof. Let $S(w)$ denote the Schubert variety in G/B associated to the element w of the Weyl group. Lemma 19 implies

$$Z \cap Y = \bigcup_{\substack{v \in W_I \\ l(wv) = l(w) + l(v)}} \overline{[\emptyset, xv, wv]} \cong \bigcup_{\substack{v \in W_I \\ l(wv) = l(w) + l(v)}} S(xvw_0) \times S(wv).$$

The restriction to Y of the line bundle $\mathcal{L}_\mu$ is the line bundle $\mathcal{L}_{-w_0\mu}^{(G/B)} \boxtimes \mathcal{L}_\mu^{(G/B)}$ on $G/B \times G/B$. Dabrowski shows in [8] that

$$H^0\big(S(w), \mathcal{L}_\mu^{(G/B)}\big) \neq 0 \quad \Leftrightarrow \quad \mu \in \alpha^+ \text{ for all } \alpha \in \Delta \text{ such that } w\alpha \in \Phi^-.$$

Assume $H^0(Z \cap Y, \mathcal{L}_\mu) \neq 0$. Then there is a $v \in W_I$ such that $l(wv) = l(w) + l(v)$, $H^0(S(xvw_0), \mathcal{L}^{(G/B)}_{-w_0\mu}) \neq 0$, and $H^0(S(wv), \mathcal{L}^{(G/B)}_{\mu}) \neq 0$. We have

$$\begin{aligned} H^0\big(S(wv), \mathcal{L}^{(G/B)}_{\mu}\big) \neq 0 \quad &\Leftrightarrow \quad \mu \in \alpha^+ \ \forall \alpha \in \Delta\colon\ wv\alpha \in \Phi^-, \\ H^0\big(S(xvw_0), \mathcal{L}^{(G/B)}_{-w_0\mu}\big) \neq 0 \quad &\Leftrightarrow \quad -w_0\mu \in \alpha^+ \ \forall \alpha \in \Delta\colon\ xvw_0\alpha \in \Phi^- \\ &\Leftrightarrow \quad -w_0\mu \in (-w_0\alpha)^+ \ \forall \alpha \in \Delta\colon\ xvw_0(-w_0\alpha) \in \Phi^- \\ &\Leftrightarrow \quad \mu \in \alpha^+ \ \forall \alpha \in \Delta\colon\ xv\alpha \in \Phi^+. \end{aligned}$$

Let $i \in I$. If $v\alpha_i \in \Phi^-$, then $wv\alpha_i \in \Phi^-$, because $l(wv) = l(w) + l(v)$. If $v\alpha_i \in \Phi^+$, then $xv\alpha_i \in \Phi^+$, because $v \in W_I$ and $x \in W^I$. This shows that $H^0(Z \cap Y, \mathcal{L}_\mu) \neq 0$ implies $\mu \in \alpha_i^+$ for all $i \in I$. $\square$

Let $\vec{n} = (n_1, \ldots, n_l) \in \mathbb{N}_0^l$, where $n_i = 0$ for all $i \notin I$. Every non-empty open subset U of Z meets the $G \times G$-orbit X_I° on which $\sigma^{\vec{n}}$ is non-zero, so the multiplication by

$$\sigma^{\vec{n}} : H^0(U, \mathcal{L}_{-\vec{n}\vec{\alpha}}) \to H^0(U, \mathcal{O}_Z)$$

is an injective map for every $U \subseteq Z$ open. Define an ideal sheaf $\sigma^{\vec{n}}\mathcal{L}_{-\vec{n}\vec{\alpha}}$ of $\mathcal{O}_Z$ by

$$\big(\sigma^{\vec{n}}\mathcal{L}_{-\vec{n}\vec{\alpha}}\big)(U) = \sigma^{\vec{n}}\big|_U \cdot \mathcal{L}_{-\vec{n}\vec{\alpha}}(U) \subseteq H^0(U, \mathcal{L}_{\vec{n}\vec{\alpha}} \otimes_{\mathcal{O}(Z)} \mathcal{L}_{-\vec{n}\vec{\alpha}}) = H^0(U, \mathcal{O}_Z)$$

for every open set U in Z.

Lemma 21. *Let $\mathcal{I}$ be the ideal sheaf of $Z \cap Y$ in $\mathcal{O}_Z$.*

(1) *$\mathcal{I}$ is generated by $\sigma_{i_1}, \ldots, \sigma_{i_r}$, i.e.*

$$\mathcal{I} = \sum_{\substack{\vec{n} \in \mathbb{N}_0^l \\ n_i = 0\ \forall i \notin I}} \sigma^{\vec{n}}\mathcal{L}_{-\vec{n}\vec{\alpha}}.$$

(2) *$(\sigma_{i_1}, \ldots, \sigma_{i_r})$ form a regular sequence in $\mathcal{O}_Z$.*
(3) *For all $n \in \mathbb{N}$ we have*

$$\mathcal{I}^n / \mathcal{I}^{n+1} \cong \bigoplus_{\substack{|\vec{n}| = n \\ n_i = 0\ \forall i \notin I}} \sigma^{\vec{n}}\mathcal{L}_{-\vec{n}\vec{\alpha}}|_{Z \cap Y}.$$

Proof. (1) $\sigma_1, \ldots, \sigma_l$ generate the ideal sheaf $\mathcal{I}_Y$ of Y in $\mathcal{O}_X$ (see e.g. [4] before Corollary 4). All σ_i are $G \times G$-invariant, so

$$\sigma_i|_Z = 0 \quad \Leftrightarrow \quad \sigma_i|_{X_I} = 0 \quad \Leftrightarrow \quad i \notin I.$$

As the scheme theoretical intersection $Z \cap Y$ is reduced by Corollary 18, $\mathcal{I}$ is generated by those σ_i where $i \in I$.

(2) To prove this assertion, the proof of Corollary 4 in [4] can be adapted. For $1 < j \leqslant r$ define

$$Z_j := Z \cap X_{\{i_j,\dots,i_r\}} = Z \cap \bigcap_{k=1}^{j-1} S_{i_k}.$$

Then we have $\mathcal{O}_{Z_j} = \mathcal{O}_Z/(\sigma_{i_1},\dots,\sigma_{i_{j-1}})$, because S_{i_k} is the divisor corresponding to σ_{i_k}. Corollary 18 assures that Z_j is reduced. By Lemma 19, the irreducible components of Z_j are $Z_{j,v} := \overline{[J, xv, wv]}$ where $J = \{i_j,\dots,i_r\}$ and $v \in W_I \cap W^J$ such that $l(wv) = l(w) + l(v)$. As none of these irreducible components is completely contained in S_{i_j}, the restriction of σ_{i_j} to $Z_{j,v}$ does not vanish for any v.

Let $f \in \mathcal{O}_Z(Z)$ such that $\sigma_{i_j} \cdot f = 0$. Then in particular $\sigma_{i_j} \cdot f|_{Z_{j,v}} = 0$ holds for the restriction. As $Z_{j,v}$ is irreducible and reduced, $\mathcal{O}_Z(Z_{j,v})$ is an integral domain. But $\sigma_{i_j}|_{Z_{j,v}} \neq 0$, so $f|_{Z_{j,v}} = 0$. This implies $f = 0$, and σ_{i_j} is no zero divisor in $\mathcal{O}_{Z_j} = \mathcal{O}_Z/(\sigma_{i_1},\dots,\sigma_{i_{j-1}})$.

(3) Because of (1) we get

$$\mathcal{I}^n = \sum_{\vec{n}} \sigma^{\vec{n}} \mathcal{L}_{-\vec{n}\vec{\alpha}},$$

where the sum is over all $\vec{n} = (n_1,\dots,n_l) \in \mathbb{N}_0^l$ such that $\sum_{i=1}^l n_i \geqslant n$ and $n_i = 0$ for all $i \notin I$. It follows directly form this that

$$\mathcal{I}^n/\mathcal{I}^{n+1} \cong \bigoplus_{\vec{n}} \sigma^{\vec{n}} \mathcal{L}_{-\vec{n}\vec{\alpha}}|_{Z\cap Y}. \qquad \square$$

Using the ideal sheaf $\mathcal{I}$ from the last lemma, a filtration of the $\tilde{B} \times \tilde{B}$-module $H^0(Z, \mathcal{L}_\lambda)$ can be defined. Indeed, the $\tilde{B} \times \tilde{B}$-modules

$$F_n := H^0\big(Z, \mathcal{L}_\lambda \otimes \mathcal{I}^n\big), \quad \text{where } n \in \mathbb{N}_0,$$

form a finite descending filtration of $H^0(Z, \mathcal{L}_\lambda)$. For $\vec{n} = (n_1,\dots,n_l) \in {\mathbb{N}_0}^l$, where $n_i = 0$ for all $i \notin I$, the multiplication by

$$\sigma^{\vec{n}} : H^0(Z, \mathcal{L}_{\lambda-\vec{n}\vec{\alpha}}) \to H^0(Z, \mathcal{L}_\lambda)$$

is injective. As $\mathcal{I}$ is generated by those σ_i where $i \in I$, and all σ_i are invariant under $\tilde{G} \times \tilde{G}$ and therefore in particular under $\tilde{B} \times \tilde{B}$,

$$F_{\vec{n}} := \operatorname{Im}\big(\sigma^{\vec{n}}\big)$$

is a $\tilde{B} \times \tilde{B}$-submodule of F_n, where $n = |\vec{n}|$.

Theorem 22.

$$F_n = \sum_{|\vec{n}|=n} F_{\vec{n}},$$

$$\operatorname{gr}_n H^0(Z, \mathcal{L}_\lambda) = F_n/F_{n+1} \cong \bigoplus_{\substack{\mu \leqslant \lambda \text{ dom.} \\ |\lambda-\mu|=n}} H^0(Z \cap Y, \mathcal{L}_\mu).$$

Proof. The short exact sequence of sheaves on Z

$$0 \to \mathcal{L}_\lambda \otimes \mathcal{I}^{n+1} \to \mathcal{L}_\lambda \otimes \mathcal{I}^n \to \mathcal{L}_\lambda \otimes \mathcal{I}^n/\mathcal{I}^{n+1} \to 0$$

induces a long exact cohomology sequence

$$0 \to F_{n+1} \to F_n \to H^0\big(Z, \mathcal{L}_\lambda \otimes \mathcal{I}^n/\mathcal{I}^{n+1}\big) \to \cdots,$$

which implies the inclusion

$$\operatorname{gr}_n H^0(Z, \mathcal{L}_\lambda) = F_n/F_{n+1} \hookrightarrow H^0\big(Z, \mathcal{L}_\lambda \otimes \mathcal{I}^n/\mathcal{I}^{n+1}\big).$$

Using the last lemma, we get

$$H^0\big(Z, \mathcal{L}_\lambda \otimes \mathcal{I}^n/\mathcal{I}^{n+1}\big) = \bigoplus_{|\vec{n}|=n} \sigma^{\vec{n}} H^0(Z \cap Y, \mathcal{L}_{\lambda-\vec{n}\vec{\alpha}}).$$

Here the sum is over all $\vec{n} = (n_1, \ldots, n_l) \in \mathbb{N}_0^l$ such that $n_i = 0$ for all $i \notin I$. Denote $\lambda - \vec{n}\vec{\alpha}$ by μ, and let $j \notin I$. Then

$$\langle \mu, \check{\alpha}_j \rangle = \Big\langle \lambda - \sum_{i \in I} n_i \alpha_i, \check{\alpha}_j \Big\rangle = \underbrace{\langle \lambda, \check{\alpha}_j \rangle}_{\geqslant 0, \text{ because } \lambda \in \Lambda^+} - \sum_{i \in I} n_i \underbrace{\langle \alpha_i, \check{\alpha}_j \rangle}_{\leqslant 0, \text{ because } i \neq j} \geqslant 0.$$

So either $\mu \in \Lambda^+ = \bigcap_{i=1}^l \alpha_i^+$ holds or $\mu \notin \bigcap_{i \in I} \alpha_i^+$. But in the second case $H^0(Z \cap Y, \mathcal{L}_\mu) = 0$ by Lemma 20. Thus, we get

$$H^0\big(Z, \mathcal{L}_\lambda \otimes \mathcal{I}^n/\mathcal{I}^{n+1}\big) = \bigoplus_{\substack{\mu \leqslant \lambda \text{ dom.} \\ |\lambda-\mu|=n}} \sigma^{(\lambda-\mu)} H^0(Z \cap Y, \mathcal{L}_\mu).$$

Altogether, there is an inclusion

$$\operatorname{gr}_n H^0(Z, \mathcal{L}_\lambda) \hookrightarrow \bigoplus_{\substack{\mu \leqslant \lambda \text{ dom.} \\ |\lambda-\mu|=n}} H^0(Z \cap Y, \mathcal{L}_\mu).$$

Consider $\vec{n} = (n_1, \ldots, n_l) \in \mathbb{N}_0^l$ with $n_i = 0$ for all $i \notin I$. As the multiplication by $\sigma^{\vec{n}} : H^0(Z, \mathcal{L}_{\lambda-\vec{n}\vec{\alpha}}) \to H^0(Z, \mathcal{L}_\lambda)$ is an injective map, its image $F_{\vec{n}}$ is isomorphic to $H^0(Z, \mathcal{L}_\mu)$ where $\mu = \lambda - \vec{n}\vec{\alpha}$. Identifying $\sum_{|\vec{n}|=n} H^0(Z, \mathcal{L}_{\lambda-\vec{n}\vec{\alpha}})$ with $\sum_{|\vec{n}|=n} F_{\vec{n}} \subseteq H^0(Z, \mathcal{L}_\lambda)$, we get the well-defined restriction map

$$\sum_{|\vec{n}|=n} H^0(Z, \mathcal{L}_{\lambda-\vec{n}\vec{\alpha}}) \to \bigoplus_{|\vec{n}|=n} H^0(Z \cap Y, \mathcal{L}_{\lambda-\vec{n}\vec{\alpha}}),$$

because any element, that is contained in $\mathrm{Im}(\sigma^{\vec{n}})$ for at least two different $\vec{n}$ such that $|\vec{n}| = n$, restricts to zero on Y. For any dominant weight $\mu \in \Lambda^+$ the restriction map $H^0(Z, \mathcal{L}_\mu) \to H^0(Z \cap Y, \mathcal{L}_\mu)$ is surjective by Corollary 18. If $\mu = \lambda - \sum_{i \in I} n_i \alpha_i$ is not dominant, then there is an index $i \in I$ such that $\mu \notin \alpha_i^+$. In this case Lemma 20 implies $H^0(Z \cap Y, \mathcal{L}_\mu) = 0$. This yields the commutative diagram

$$\begin{tikzcd}
\sum_{|\vec{n}|=n} F_{\vec{n}} \arrow[rr, "\simeq"] \arrow[d, hook] & & \sum_{|\vec{n}|=n} H^0(Z, \mathcal{L}_{\lambda-\vec{n}\vec{\alpha}}) \arrow[d, two heads] \\
F_n \arrow[r, two heads] & F_n/F_{n+1} = \mathrm{gr}_n H^0(Z, \mathcal{L}_\lambda) \arrow[r, hook] & \bigoplus_{\substack{\mu \leqslant \lambda \text{ dom.} \\ |\lambda-\mu|=n}} H^0(Z \cap Y, \mathcal{L}_\mu).
\end{tikzcd}$$

As this map is surjective, the first part of the assertion follows. Now $F_n/F_{n+1} \cong \bigoplus_\mu H^0(Z \cap Y, \mathcal{L}_\mu)$ implies

$$F_n \cong \bigoplus_{\mu} H^0(Z \cap Y, \mathcal{L}_\mu) \oplus F_{n+1}.$$

As $\sum F_{\vec{n}}$ and F_{n+1} are both submodules of F_n and the above map $\sum F_{\vec{n}} \to \bigoplus H^0(Z \cap Y, \mathcal{L}_\mu)$ is surjective, we get

$$F_n = \sum_{|\vec{n}|=n} F_{\vec{n}} + F_{n+1}.$$

Iterating the last steps gives

$$F_n = \sum_{|\vec{n}|=n} F_{\vec{n}} + \sum_{|\vec{n}|=n+1} F_{\vec{n}} + F_{n+2} = \sum_{|\vec{n}|=n} F_{\vec{n}} + F_{n+2},$$

because $F_{\vec{m}} \subseteq F_{\vec{n}}$ if $m = (m_1, \ldots, m_l)$ and $n = (n_1, \ldots, n_l)$ such that $m_i \geqslant n_i$ for all $1 \leqslant i \leqslant l$. As the filtration is finite, the second part of the assertion follows by induction. □

Corollary 23. *The set $\mathcal{M}_Z^{(\lambda)}$ is a basis of $H^0(Z, \mathcal{L}_\lambda|_Z)$.*

Proof. By Proposition 14 the set $\mathcal{M}_Z^{(\lambda)}$ is linearly independent. Theorem 22 implies

$$\begin{aligned}
\dim H^0(Z, \mathcal{L}_\lambda|_Z) &= \dim \mathrm{gr} H^0(Z, \mathcal{L}_\lambda|_Z) \\
&= \dim \bigoplus_{\mu \leqslant \lambda \text{ dom.}} H^0(Z \cap Y, \mathcal{L}_\mu|_{Z \cap Y}) \\
&= \sum_{\mu \leqslant \lambda \text{ dom.}} \left|\left\{p_\pi^{(\mu)} \text{ standard on } Z \cap Y\right\}\right| \\
&= \left|\mathcal{M}_Z^{(\lambda)}\right|. \qquad \square
\end{aligned}$$

Remark 24. This statement cannot be generalized to $X = \overline{G/H}$. In general, the set $\mathcal{M}_Z^{(\lambda)}$ described in Remark 15 is not a basis of $H^0(Z, \mathcal{L}_\lambda)$. In [1], the following counter-example is given: Consider the group $G = PSL(n+1)$ over $k = \mathbb{C}$ and the involution $\theta : G \to G$, $g \mapsto (g^{-1})^t$. These lead to the symmetric space $G/H = PSL(n+1)/PSO(n+1)$ with wonderful compactification $X = \overline{G/H}$. Let Z be a B-stable divisor that is not G-stable, i.e. Z is the closure of a B-orbit of codimension 1 in the dense G-orbit. It can be shown, that for any regular dominant weight λ the restriction map $H^0(X, \mathcal{L}_\lambda) \to H^0(Z, \mathcal{L}_\lambda)$ is surjective. But in $H^0(Z, \mathcal{L}_\lambda)$ there are weight vectors of weights that do not appear in $\mathcal{M}_Z^{(\lambda)}$. So the set $\mathcal{M}_Z^{(\lambda)}$ does not generate $H^0(Z, \mathcal{L}_\lambda)$.

6. Standard monomials

Let $\lambda \in \Lambda^+$ and consider the $\tilde{B} \times \tilde{B}$-module $H^0(X, \mathcal{L}_\lambda)$. The aim of this section is to construct a basis of this module that—like classical standard monomials—has the following properties:

(1) The elements of the basis are indexed by the set of LS-paths $\bigcup_{\mu \leqslant \lambda \,\mathrm{dom.}} B_\mu$. We call them path vectors. They are weight vectors whose weight is determined by the end point of the corresponding path.
(2) Let Z be the closure of a $B \times B$-orbit in X. The restriction to Z of those path vectors which are standard on Z with respect to λ form a basis of $H^0(Z, \mathcal{L}_\lambda)$.
(3) Let Z be the closure of a $B \times B$-orbit in X. The restriction to Z of those path vectors which are not standard on Z with respect to λ form a basis of the kernel of the restriction map $H^0(X, \mathcal{L}_\lambda) \to H^0(Z, \mathcal{L}_\lambda)$.

Definition 25. Let $\lambda, \mu \in \Lambda^+$ be dominant weights and Z the closure of the $B \times B$-orbit $[I, x, w]$ in X. The LS-path $\pi \in B_\mu$ and the corresponding path vector are called *standard on Z with respect to* λ if π is standard on Z and $\mu \leqslant \lambda$ such that $\mu = \lambda - \sum_{i=1}^{l} n_i \alpha_i \in \Lambda^+$ where $n_i = 0$ for all $i \notin I$.

The set $\mathcal{M}^{(\lambda)} = \{\sigma^{(\lambda-\mu)} x_\pi^{(\mu)} \mid \mu \in \Lambda^+,\ \mu \leqslant \lambda,\ \pi \in B_\mu\}$ defined in Section 4 is a basis of $H^0(X, \mathcal{L}_\lambda)$ that is compatible with the restriction to $B \times B$-orbit closures. It fulfills the first two properties. This is true for arbitrary continuations $x_\pi^{(\mu)} \in H^0(X, \mathcal{L}_\mu)$ of the standard monomials $p_\pi^{(\mu)} \in H^0(Y, \mathcal{L}_\mu)$.

Now choose for any $\lambda \in \Lambda^+$ and $\pi \in B_\lambda$ an extension $x_\pi^{(\lambda)}$ of the standard monomial $p_\pi^{(\lambda)} \in H^0(Y, \mathcal{L}_\lambda)$ to X. In general, the chosen set $\mathcal{M}^{(\lambda)} = \{\sigma^{(\lambda-\mu)} x_\pi^{(\mu)} \mid \mu \in \Lambda^+,\ \mu \leqslant \lambda,\ \pi \in B_\mu\}$ does not have property (3), because the restriction of $\sigma^{(\lambda-\mu)} x_\pi^{(\mu)}$ to a $B \times B$-orbit on which π is not standard does not need to be zero. But starting from those, new standard monomials $\sigma^{(\lambda-\mu)} y_\pi^{(\mu)}$ can be constructed as linear combinations of the $\sigma^{(\lambda-\mu)} x_\pi^{(\mu)}$, which have all three properties.

Theorem 26. *Let $\lambda \in \Lambda^+$ be a dominant weight. For each $\pi \in B_\lambda$ there is a global section $y_\pi^{(\lambda)} \in H^0(X, \mathcal{L}_\lambda)$ such that $y_\pi^{(\lambda)}|_Y = p_\pi^{(\lambda)}$ and $y_\pi^{(\lambda)}|_Z = 0$ for all $B \times B$-orbit closures Z on which π is not standard.*

Proof. The claim is proven by constructing $y_\pi^{(\lambda)}$ recursively for all λ. Let $\lambda \in \Lambda^+$. Assume that $y_\nu^{(\mu)}$, where $\nu \in B_\mu$, is already constructed for all dominant $\mu < \lambda$. Take a path $\pi \in B_\lambda$ and consider

$$\hat{Z}_\pi := \bigcup_{\substack{\pi \text{ not standard} \\ \text{on } \overline{[I,x,w]}}} [I, x, w] \subseteq X.$$

If π is not standard on the closure $\overline{[I, x, w]}$ of a $B \times B$-orbit $[I, x, w]$, it is not standard on the closure of any $B \times B$-orbit contained in $\overline{[I, x, w]}$ either. So $\hat{Z}_\pi$ is closed and $B \times B$-stable. Its irreducible components $Z_1, \dots, Z_t$ are closures of $B \times B$-orbits. The restriction of $x_\pi^{(\lambda)}$ to each Z_i is a linear combination of elements in $\mathcal{M}_{Z_i}^{(\lambda)}$ of the same weight. Hence, there are coefficients $\alpha_{i\nu}, \beta_{i\nu} \in k$ such that

$$x_\pi^{(\lambda)}\big|_{Z_i} = \sum_{\substack{\nu \in B_\lambda \\ \nu \text{ standard on } Z_i \\ \nu(1)=\pi(1)}} \alpha_{i\nu} x_\nu^{(\lambda)}\big|_{Z_i} + \sum_{\substack{\mu<\lambda \text{ dom.} \\ \nu \in B_\mu \\ \nu \text{ standard on } Z_i \text{ w.r.t. } \lambda \\ \nu(1)=\pi(1)}} \beta_{i\nu} \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{Z_i}.$$

This yields for the restriction to $Z_i \cap Y$

$$\begin{array}{ccc} x_\pi^{(\lambda)}|_{Z_i \cap Y} & = & \sum \alpha_{i\nu} x_\nu^{(\lambda)}|_{Z_i \cap Y} \\ \| & & \| \\ p_\pi^{(\lambda)}|_{Z_i \cap Y} & = & \sum \alpha_{i\nu} p_\nu^{(\lambda)}|_{Z_i \cap Y}. \end{array}$$

As π is not standard on $Z_i \cap Y$, we have $p_\pi^{(\lambda)}|_{Z_i \cap Y} = 0$. But the restrictions $p_\nu^{(\lambda)}|_{Z_i \cap Y}$ form a basis of $H^0(Z_i \cap Y, \mathcal{L}_\lambda)$, thus $\alpha_{i\nu} = 0$ for all ν.

In case $\lambda \in \Lambda^+$ is minimal with respect to the order $\leqslant$, that means there is no dominant $\mu < \lambda$, this implies that every extension of $p_\pi^{(\lambda)}$ to X has the required properties. Actually, we have an isomorphism $H^0(X, \mathcal{L}_\lambda) \cong H^0(Z, \mathcal{L}_\lambda)$, so the choice of the extension is canonical. Denote this extension by $y_\pi^{(\lambda)}$.

For λ not minimal, the equation

$$x_\pi^{(\lambda)}\big|_{Z_i} = \sum_{\substack{\mu<\lambda \text{ dom.} \\ \nu \in B_\mu \\ \nu \text{ standard on } Z_i \text{ w.r.t. } \lambda \\ \nu(1)=\pi(1)}} \beta_{i\nu} \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{Z_i}$$

remains. The following argument shows that the coefficients $\beta_{i\nu}$ may be chosen in such a way that $\beta_{i\nu} = \beta_{j\nu}$ for all $i, j \in \{1, \dots, t\}$. If ν is not standard on Z_i, then $y_\nu^{(\mu)}|_{Z_i} = 0$, and $\beta_{i\nu}$ can be chosen arbitrarily. But each ν is standard on at least one irreducible component Z_i. If ν is standard on two irreducible components Z_i and Z_j, then it is standard on their intersection $Z_i \cap Z_j$ as well. This fact is a generalization of the analogous fact for Schubert varieties and it is proved subsequently in Lemma 28. We have

$$x_\pi^{(\lambda)}\big|_{Z_i \cap Z_j} = \sum_{\substack{\nu \text{ standard on } Z_i \\ \text{w.r.t. } \lambda}} \beta_{i\nu} \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{Z_i \cap Z_j} = \sum_{\substack{\nu \text{ standard on } Z_j \\ \text{w.r.t. } \lambda}} \beta_{j\nu} \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{Z_i \cap Z_j}.$$

As $\sigma^{(\lambda-\mu)} y_\nu^{(\mu)}|_{Z_i \cap Z_j} = 0$ for all ν that are not standard on $Z_i \cap Z_j$ with respect to λ, this implies

$$\sum_{\substack{\nu \text{ standard on } Z_i \cap Z_j \\ \text{w.r.t. } \lambda}} \beta_{i\nu} \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{Z_i \cap Z_j} = \sum_{\substack{\nu \text{ standard on } Z_i \cap Z_j \\ \text{w.r.t.} \lambda}} \beta_{j\nu} \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{Z_i \cap Z_j}.$$

But all appearing $\sigma^{(\lambda-\mu)} y_\nu^{(\mu)}|_{Z_i \cap Z_j}$ are linearly independent. Therefore we get $\beta_{i\nu} = \beta_{j\nu}$. Defining $\beta_\nu := \beta_{1\nu} = \cdots = \beta_{t\nu}$ leads to

$$x_\pi^{(\lambda)}\big|_{\hat{Z}_\pi} = \sum \beta_\nu \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{\hat{Z}_\pi}.$$

Now take

$$y_\pi^{(\lambda)} := x_\pi^{(\lambda)} - \sum \beta_\nu \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}.$$

This yields for the restriction to Y

$$y_\pi^{(\lambda)}\big|_Y = x_\pi^{(\lambda)}\big|_Y - 0 = p_\pi^{(\lambda)}$$

and for the restriction to $\hat{Z}_\pi$

$$y_\pi^{(\lambda)}\big|_{\hat{Z}_\pi} = x_\pi^{(\lambda)}\big|_{\hat{Z}_\pi} - \sum \beta_\nu \sigma^{(\lambda-\mu)} y_\nu^{(\mu)}\big|_{\hat{Z}_\pi} = 0. \qquad \square$$

Corollary 27. *Let $\lambda \in \Lambda^+$. The set*

$$\mathcal{S}^{(\lambda)} := \left\{\sigma^{(\lambda-\mu)} y_\nu^{(\mu)} \;\middle|\; \mu \in \Lambda^+,\ \mu \leqslant \lambda,\ \pi \in B_\mu\right\}$$

has properties (1)–(3).

To complete the proof of Proposition 26, it remains to show

Lemma 28. *Let Z_1, Z_2 be irreducible components of $\hat{Z}_\pi$ where $\pi \in B_\lambda$. If $\mu < \lambda$ is dominant and $\nu \in B_\mu$ is standard on Z_1 and Z_2 with respect to λ, then ν is also standard on $Z_1 \cap Z_2$ with respect to λ.*

Proof. Let $Z_1 = \overline{[I_1, x_1, w_1]}$, $Z_2 = \overline{[I_2, x_2, w_2]}$, and $\lambda - \mu = \sum n_k \alpha_k$. From $\nu \in B_\mu$ standard on Z_i follows $n_k = 0\ \forall k \notin I_i$, hence $n_k = 0\ \forall k \notin I_1 \cap I_2$. As we have $Z_1 \cap Z_2 \subseteq X_{I_1 \cap I_2}$, the weight μ has the property stated in Definition 25.

By definition, the path $\nu \in B_\mu$ is standard on Z_i if and only if ν is standard on $Z_i \cap Y$. It remains to show that a path which is standard on two Schubert varieties in $G/B \times G/B$ is also standard on their intersection.

Let $\nu \in B_\mu$ be standard on $Y_i = \overline{[\emptyset, x_i, w_i]}$, $i = 1, 2$. We claim that in that case ν is also standard on $Y_1 \cap Y_2$. Indeed, ν is standard on $Y = \overline{[\emptyset, x, w]} \cong S(xw_0) \times S(w)$ if and only if $i(\nu) \leqslant (xw_0, w)$, where $i(\nu)$ is the initial direction of the path $i(\nu)$. Denoting $i(\nu) = (\tilde{x}w_0, \tilde{w})$, ν is standard on Y if $\tilde{x} \geqslant x$ and $\tilde{w} \leqslant w$. In particular, ν is standard on $\overline{[\emptyset, \tilde{x}, \tilde{w}]}$,

$$\begin{aligned}
\nu \text{ standard on } Y_1 \text{ and } Y_2 \quad &\Leftrightarrow \quad \tilde{x} \geqslant x_1,\ \tilde{x} \geqslant x_2,\ \tilde{w} \leqslant w_1,\ \tilde{w} \leqslant w_2 \\
&\Leftrightarrow \quad [\emptyset, \tilde{x}, \tilde{w}] \leqslant [\emptyset, x_1, w_1] \text{ and } [\emptyset, \tilde{x}, \tilde{w}] \leqslant [\emptyset, x_2, w_2] \\
&\Leftrightarrow \quad \overline{[\emptyset, \tilde{x}, \tilde{w}]} \subseteq Y_1 \cap Y_2 \\
&\Rightarrow \quad \nu \text{ standard on } Y_1 \cap Y_2. \qquad \square
\end{aligned}$$

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